On 2-Colorability Problem for Hypergraphs with P_8-free Incidence Graphs

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Abstract: *A 2-coloring of a hypergraph is a mapping from its vertex set to a set of two colors such that no edge is monochromatic. The hypergraph 2- Coloring Problem is the question whether a given hypergraph is 2-colorable. It is known that deciding the 2-colorability of hypergraphs is NP-complete even for hypergraphs whose hyperedges have size at most 3. In this paper, we present a polynomial time algorithm for deciding if a hypergraph, whose incidence graph is P_8-free and has a dominating set isomorphic to C_8, is 2-colorable or not. This algorithm is semi generalization of the 2-colorability algorithm for hypergraph, whose incidence graph is P_7-free presented by Camby and Schaudt.*

Keywords: *Hypergraph, Dominating set, P_k-free graph, Computational Complexity.*

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1. Introduction

A pair $H = (V, E)$ is a (finite) hypergraph if V is a finite vertex set and E is a collection of subsets of V called the hyperedges of H. Hypergraphs are a natural generalization of undirected graphs; unlike edges, hyperedges are not necessarily two-elementary.

A hypergraph $H = (V, E)$ is 2-colorable if its vertex set V has a partition $V = V_1 \cup V_2$ such that every hyperedge $e \in E$ has at least one vertex from each of the sets V_1 and V_2. The hypergraph 2- Coloring Problem (also called Bicoloring Problem, Set Splitting Problem in [8]) is the question whether a given hypergraph is 2-colorable.

The property of 2-colorability was introduced and studied by Bernstein [4] in the early 1900s for infinite hypergraphs. The 2-colorability of finite hypergraphs has been studied for about ninety years due to its applications in theoretical computer science, see for example [2, 6, 7, 12]), as well as in practical computer science, especially in wireless networks [16].

If every hyperedge is of size 2, i.e., for graphs, the problem is well understood, since graph 2-colorability is equivalent to having no odd cycle. Excluding this special case, though, much less is known and deciding the 2-colorability of hypergraphs is NP-complete even for hypergraphs whose hyperedges have size at most 3 [11]. Another proof of this result is given in [10] using a nice reduction from the Satisfiability Problem SAT to the Hypergraph 2-Coloring Problem.

Several fundamental approaches in hypergraph 2-coloring appeared in the literature. They are related to the various types of constraints that are imposed on the hyperedges while coloring the vertices. One of these approaches is the 2-colarability problem of k-uniform hypergraph, i.e., every hyperedge is of fixed size $k \geq 2$. A line of research (e.g., [12]) has been devoted to extremal problems asking for the least number of hyperedges that an k-uniform hypergraph can have without being 2-colorable. In the same direction, some sufficient conditions for the existence of a 2-coloring of k-uniform hypergraphs have been found (e.g. [15]). The degree of vertices of k-uniform hypergraph is taking into consideration also in studying this problem. The degree of a vertex v in a hypergraph H is the number of hyperedges of H which contain v. In this approache, a study of the complexity of 2-coloring in k-uniform hypergraphs of high minimum degree is given in [13]. The 2-coloring in k-regular k-uniform hypergraphs (i.e. the degree of every vertex is k) is extensively studied in [1, 9].

Another direction of investigation is to look to a special structure of the incidence graph associated with a hypergraph. The incidence graph of a hypergraph $H = (V, E)$ is the bipartite graph $G = (V \cup E, I)$ where $v \in V$ and $e \in E$ are adjacent (i.e. $ve \in I$) if and only if $v \in e$. Recently, van't Hof and Paulusma [14] show that hypergraph 2-colorability is solvable in polynomial time for hypergraphs with P_6-free incidence graphs. This result is extended in [5] by Camby and Schaudt for hypergraphs with P_7-free incidence graphs.

The purpose of this paper is to solve in polynomial time the 2-colarability problem for hypergraphs with P_8-free incidence graphs whose dominated set is C_8 (see Figure 1).

The rest of this section contains the notions and tools used in our algorithm. Section 2, presents the recognition of different cases that can be occurs in our treatment for this problem. The complete algorithm and its complexity are discussed in section 3. Section 4 is the conclusion and future work.

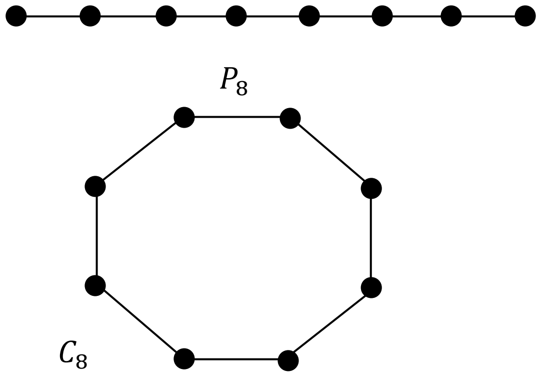

Figure 1. The forbidden configuration P_8 and the dominated configuration C_8.

Let P_k be the induced path on k vertices and let C_k be the induced cycle on k vertices. If G and H are two graphs, we say that G is H-free if H does not appear as an induced subgraph of G. A dominating set of a graph G is a vertex subset D such that every vertex not in D has a neighbor in D. A connected dominating set of a graph G is a dominating set D whose induced subgraph, henceforth denoted $G[D]$, is connected. A characterization of P_k-free graph in term of dominating sets is given in the following theorem.

Theorem 1 [5] *Let G be a graph and $k \geq 4$. The following assertions are equivalent.*

1) *G is P_k-free.*
2) *Every connected induced subgraph H of G admits a connected dominating set Dsuch that $H[D]$ is P_{k-2}-free or $H[D]$ is isomorphic to C_k.*

Let G be a connected P_k-free graph, $k \geq 4$, on n vertices and m edges. Camby and Schaudt in [5] show that the computation of a connected dominating set D such that $G[D]$ is P_{k-2}-free or $G[D]$ is isomorphic to C_k can be done in time $O(n^5(n+m))$.

Let $H = (V, E)$ be a hypergraph. We denote by (A, B) to a 2-coloring of H, that is, A, B are non-empty subset of V, $A \cup B = V. A \cap B = \emptyset$, and for every hyperedge $e \in E$, $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$. Since we are searching for a 2-coloring, hyperedges containing exactly one vertex are excluded. Moreover, if no hyperedge $e \in E$ is properly contained in another hyperedge $e' \in E$ then H is called a Sperner family or clutter. In the database community (see e.g., [3]), clutters are called reduced hypergraphs. The following observation was proven in [14] and in [5]. In order to be self-contained, we give a quick proof of it.

Lemma 1 *H can be assumed a clutter*.

Proof Let $e, f \in E$ such that $e \subseteq f$. We claim that H is 2-colorable if and only if $H' = (V, E - \{f\})$ is 2-colorable. Clearly, if H is 2-colorable then H' is 2-colorable. Let (A, B) be a 2-coloring of H'. Since $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$ and $e \subseteq f$ then $f \cap A \neq \emptyset$ and $f \cap B \neq \emptyset$, so (A, B) is a 2-coloring of H.

Observe that, if $H = (V, E)$ is a hypergraph whose incidence graph is P_8-free and if we delete for every pair $e, f \in E$ with $e \subseteq f$ the hyperedge f from H, the resulting hypergraph is a clutter and its incidence graph is still P_8-free. So, from now on, we assume that $H = (V, E)$ is a clutter whose incidence graph $G = (V \cup E, I)$ is P_8-free. Moreover, we may assume that H is connected, that is, G is connected. By Theorem 1, there is a connected dominating set D of G such that $G[D]$ is P_6-free or $G[D] \cong C_8$. In this paper, we suppose $G[D] \cong C_8$ and we leave the discussion of the case $G[D]$ is P_6-free for future work.

2. Hypergraph 2-Colorability Problem with Incidence Graph P_8-free Whose Dominating set is C_8

Through this section, the dominating set $D = \{x_1, f_1, x_2, f_2, x_3, f_3, x_4, f_4\}$ where $X = \{x_1, x_2, x_3, x_4\} \subseteq V$, $F = \{f_1, f_2, f_3, f_4\} \subseteq E$ and $G[D] = x_1 f_1 x_2 f_2 x_3 f_3 x_4 f_4 x_1 \cong C_8$. Let $R = V - X$. For a subset $J \subseteq \{1, 2, 3, 4\}$ we define $V_J = \{x \in R : x \in f_j \text{ if } j \in J\}$. In other words, V_J is the set of vertices of R that are dominated only by $f_j, j \in J$. For short, any $J \subseteq \{1, 2, 3, 4\}$ will be denoted by its elements only. For example, if $J = \{1, 2\}$ then we write $J = 12$ and $V_{12} = \{x \in R : x \in f_1 \cap f_2 \text{ and } x \notin f_3 \cup f_4\}$. Let $f \in E$, we denote to the set of vertices in X that dominate f by $d(f)$, that is $d(f) = \{x \in X : x \in f\}$. Note that, for any $f_j \in F$, $d(f_j) = \{x_j, x_{j+1}\}$ (vertex index arithmetic is modulo 4). Let's treat first some trivial cases.

Observe that if $R = \emptyset$ then H is 2-colorable if and only if $E = F$. In this case $(\{x_1, x_3\}, (x_2, x_4\})$ is a 2-coloring of H.

Suppose $R \neq \emptyset$. If E does not contain a hyperedge g such that $d(g) = \{x_1, x_3\}$ and every hyperedge h such that $d(h) = \{x_2, x_4\}$ satisfies that $d(h) \neq h$ (i.e., $h \cap R \neq \emptyset$), then $(\{x_1, x_3\} \cup R, \{x_2, x_4\})$ is a 2-coloring of H. Similarly, if E does not contain a hyperedge h such that $d(h) = \{x_2, x_4\}$ and every hyperedge g such that $d(g) = \{x_1, x_3\}$ satisfies that $d(g) \neq g$, then $(\{x_1, x_3\}, \{x_2, x_4\} \cup R)$ is a 2-coloring of H. So, we can suppose from now on that, $R \neq \emptyset$ and E contains a hyperedge g with $d(g) = \{x_1, x_3\}$ and a hyperedge h with $d(h)=\{x_2, x_4\}$.

We solve our 2-coloring problem by discussing the following cases:

1) E contains exactly one of the two hyperedges $g = \{x_1, x_3\}$ and $h = \{x_2, x_4\}$. Figure 2 illustrates an example of this case.

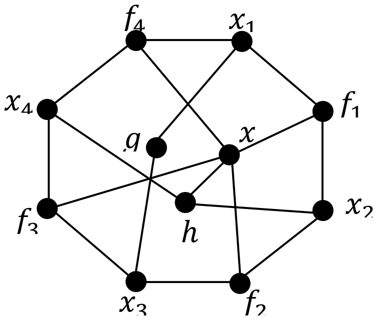

Figure 2. A hypergraph corresponding to case 1.

2) E contains both the two hyperedges $g = \{x_1, x_3\}$ and $h = \{x_2, x_4\}$. Figure 3 illustrates an example of this case

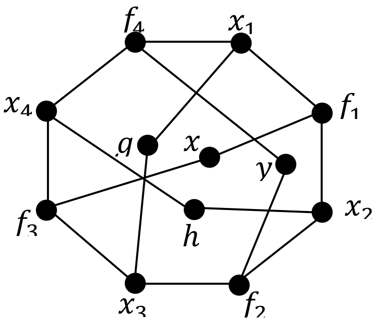

Figure 3. A hypergraph corresponding to case 2.

3) E does not contain $g = \{x_1, x_3\}$ nor $h = \{x_2, x_4\}$. Figure 4 illustrates an example of this case

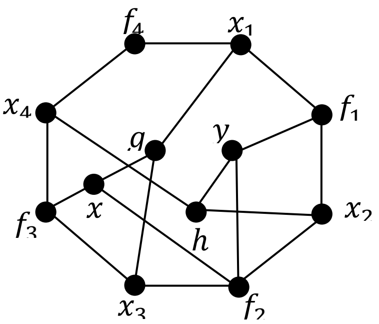

Figure 4. A hypergraph corresponding to case 3.

For this purpose, we proof a sequence of Lemmas and Theorems that discuss all relevant cases.

Lemma 2 *For every $x \in R$ there is at least two hyperedges $f_i, f_j \in F$ such that $x \in f_i \cap f_j$.*

Proof If there is $x \in R$ such that $x \in f_j$, $1 \leq j \leq 4$, and $x \notin f_{j+1} \cap f_{j+2} \cap f_{j+3}$ then $x f_j x_{j+1} f_{j+1} x_{j+2} f_{j+2} x_{j+3} f_{j+3} \cong P_8$, contradiction. □

Lemma 2 allows us to partition R into:

$$R = V_{12} \cup V_{13} \cup V_{14} \cup V_{23} \cup V_{24} \cup V_{34} \cup \bigcup_{j=1}^{4} V_{jj+1j+2} \cup V_{1234}$$

Lemma 3 *Let $x \in R$ and $g, h \in E$ such that $d(g) = \{x_1, x_3\}$ and $d(h) = \{x_2, x_4\}$.*

1) *If $x \in V_{12} \cup V_{34}$ then $x \in h$ and $x \notin g$.*
2) *If $x \in V_{14} \cup V_{23}$ then $x \in g$ and $x \notin h$.*
3) *If $x \in V_{jj+1j+2}$, $1 \leq j \leq 4$, then $x \in h$ and $x \in g$.*
4) *If $x \in V_{13} \cup V_{24}$ then either $x \in g$ and $x \in h$ or $x \notin g$ and $x \notin h$.*

Proof 1) Let $x \in V_{jj+1}$, $j = 1$ or $j = 3$. If $x \notin h$ then $h x_{j+3} f_{j+3} x_j f_j x f_{j+1} x_{j+2} \cong P_8$, contradiction. If $x \in g$ then $f_{j+2} x_{j+3} f_{j+3} x_j g x f_{j+1} x_{j+1} \cong P_8$, contradiction.
2) Similar to 1.
3) Let $x \in V_{jj+1j+2}$, $1 \leq j \leq 4$. If $x \notin g$ then, for $j = 1$ or $j = 3$, $g x_j f_j f_{j+3} x_{j+3} f_{j+2} x f_{j+1} x_{j+1} \cong P_8$, and for $j = 2$ or $j = 4$, $g x_{j+3} f_{j+3} x_j f_j x f_{j+1} x_{j+2} \cong P_8$, contradiction. If $x \notin h$ then, for $j = 1$ or $j = 3$, $h x_{j+3} f_{j+3} x_j f_j x f_{j+1} x_{j+2} \cong P_8$, and for $j = 2$ or $j = 4$, $h x_j f_j f_{j+3} x_{j+3} f_{j+2} x f_{j+1} x_{j+1} \cong P_8$, contradiction.
4) suppose $x \in g$ and $x \notin h$. If $x \in V_{13}$then $x_3 g x f_1 x_2 h x_4 f_4 \cong P_8$. If $x \in V_{24}$ then $x_3 g x f_4 x_4 h x_2 f_1 \cong P_8$, contradiction. The case when $x \notin g$ and $x \in h$ is similar. □

The following Corollaries are immediate results from Lemma 3.

Corollary 1 *If E contains $g = \{x_1, x_3\}$ and does not contain $h = \{x_2, x_4\}$ then*

$$R = V_{12} \cup V_{13} \cup V_{24} \cup V_{34} \cup V_{1234}$$

Corollary 2 *If E contains $h = \{x_2, x_4\}$ and does not contain $g = \{x_1, x_3\}$ then*

$$R = V_{13} \cup V_{14} \cup V_{23} \cup V_{24} \cup V_{1234}$$

Corollary 3 *If E contains both $h = \{x_2, x_4\}$ and $g = \{x_1, x_3\}$ then*

$$R = V_{13} \cup V_{24} \cup V_{1234}$$

Corollary 4 *Suppose E does not contain $h = \{x_2, x_4\}$ nor $g = \{x_1, x_3\}$. Let $g_1, \dots, g_k, h_1, \dots, h_r \in E$ such that for $1 \leq i \leq k$ and for $1 \leq j \leq r$ $d(g_i) = \{x_1, x_3\}$ and $d(h_j) = \{x_2, x_4\}$.*

1) *$V_{12} \cup V_{34} \subseteq \bigcap_{j=1}^{r} h_j$ and for $1 \leq j \leq k$ $(V_{12} \cup V_{34}) \cap g_j = \emptyset$.*
2) *$V_{14} \cup V_{23} \subseteq \bigcap_{i=1}^{k} g_j$ and for $1 \leq j \leq r$ $(V_{14} \cup V_{23}) \cap h_j = \emptyset$.*
3) *For $1 \leq j \leq 4$, $V_{jj+1j+2} \subseteq \bigcap_{j=1}^{k} g_j \cap \bigcap_{j=1}^{r} h_j$.*
4) *V_{13} (resp. V_{24}) can be partitioned to $\overleftarrow{V}_{13}, \overrightarrow{V}_{13}$ (resp. $\overleftarrow{V}_{24}, \overrightarrow{V}_{24}$) such that:*
   a) $\overrightarrow{V}_{13} \cup \overrightarrow{V}_{24} \subseteq \bigcap_{j=1}^{k} g_i \cap \bigcap_{j=1}^{r} h_j$ and

b) *for* $1 \leq i \leq k$, $1 \leq j \leq r$, $(\overleftarrow{V}_{13} \cup \overleftarrow{V}_{24}) \cap (g_i \cup h_j) = \emptyset$

The following two Lemmas are analogue, so we prove them together.

Lemma 3 *Let* $f \in E$ *such that* $d(f) = \{x_1, x_2, x_4\}$ *or* $d(f) = \{x_3, x_2, x_4\}$, *then* $V_{13} \cup V_{24} \subseteq f$. *In addition, if* E *contains* g *with* $d(g) = \{x_1, x_3\}$ *then* $V_{12} \cup V_{34} \subseteq f$.

Lemma 4 *Let* $f \in E$ *such that* $d(f) = \{x_2, x_1, x_3\}$ *or* $d(f) = \{x_4, x_1, x_3\}$, *then* $V_{13} \cup V_{24} \subseteq f$. *In addition, if* E *contains* h *with* $d(h) = \{x_2, x_4\}$ *then* $V_{14} \cup V_{23} \subseteq f$.

Proof The reader can check that the two Lemmas can be gathered together as following: Let $f \in E$ such that $d(f) = \{x_j, x_{j+1}, x_{j+2}\}$, $1 \leq j \leq 4$, then $V_{jj+2} \cup V_{j+1j+3} \subseteq f$. In addition, if E contains h and g with $d(h) = \{x_2, x_4\}$ and $d(g) = \{x_1, x_3\}$ then $V_{j+1j+2} \cup V_{jj+3} \subseteq f$.

Let $x \in V_{jj+2}$, and $x \notin f$ then, $f_{j+1}x_{j+1}fx_jf_{j+3}x_{j+3}f_{j+2}x \cong P_8$, let $x \in V_{j+1j+3}$, and $x \notin f$ then $f_jx_{j+1}fx_{j+2}f_{j+2}x_{j+3}f_{j+3}x \cong P_8$, contradiction. Let $x \in V_{j+1j+2} \cup V_{jj+3}$ and $x \notin f$. By Corollary 4, $x \notin h$ when $j = 1{,}3$ and $x \notin g$ when $j = 2{,}4$. If $x \in V_{j+1j+2}$ then, $xf_{j+1}x_{j+2}fx_jf_{j+3}x_{j+3}e \cong P_8$, where, $e = h$ if $j = 1{,}3$ or $e = g$ if $j = 2{,}4$, contradiction. If $x \in V_{jj+3}$ then, $xf_jx_jfx_{j+2}f_{j+2}x_{j+3}e \cong P_8$, where, $e = h$ if $j = 1{,}3$ or $e = g$ if $j = 2{,}4$, contradiction. □

Theorem 2 *Suppose* E *contains* $g = \{x_1, x_3\}$ *and does not contain* $h = \{x_2, x_4\}$. H *is 2-colorable if and only if the following conditions hold:*

1) $f_1 \cap R \neq \emptyset$ *or* $f_2 \cap R \neq \emptyset$.
2) $f_3 \cap R \neq \emptyset$ *or* $f_4 \cap R \neq \emptyset$.
3) *If* $R = \{x\}$ *then*
   a) $\{x\} = V_{1234}$.
   b) *there is at least one* $f \notin E$ *such that* $|f| = |d(f)| = 3$.

Proof By Corollary 1, $R = V_{12} \cup V_{13} \cup V_{24} \cup V_{34} \cup V_{1234}$. Let $f \in E$ such that $|f| = |d(f)| = 3$. Since H is clutter and $g \in E$, $f = \{x_1, x_2, x_4\}$ or $f = \{x_3, x_2, x_4\}$. Suppose H is 2-colorable, let (A, B) be a 2-coloring of H. Since $g \cap R = \emptyset$, we can suppose without loss of generality that $x_1 \in A, x_3 \in B$.

If $f_1 \cap R = \emptyset$ then $x_2 \in B$. If $f_2 \cap R = \emptyset$ then $f_2 \cap A = \emptyset$, contradiction. So, $f_2 \cap R$ must be non-empty. Similarly, $f_3 \cap R$ and $f_4 \cap R$ cannot be both empty.

Let $R = \{x\}$ and $\{x\} \neq V_{1234}$. By conditions 1 and 2, $\{x\} = V_{13}$ or $\{x\} = V_{24}$. Without loss of generality suppose that $\{x\} = V_{13}$. Then $x \notin f_2$ and $x \notin f_4$. So $x_2 \in A$ and $x_4 \in B$, therefore $x \in B$. But now $f_3 \cap A = \emptyset$, contradiction, so $\{x\} = V_{1234}$. Suppose that $f = \{x_1, x_2, x_4\} \in E$ and $f' = \{x_3, x_2, x_4\} \in E$. Without loss of generality suppose that $x \in A$. Since $x \notin f$, $x_2 \in B$ or $x_4 \in B$. If $x_2 \in B$ and $x_4 \in A$ then $f_4 \cap B = \emptyset$, if $x_2 \in A$ and $x_4 \in B$ then $f_1 \cap B = \emptyset$, contradiction. So, $x_2 \in B$ and $x_4 \in B$. Now, $f' \cap A = \emptyset$, contradiction.

Suppose conditions 1, 2 and 3 are hold. We will construct a 2-coloring of H. Note that, by conditions 1 and 2, R cannot be equal to V_{12} or V_{34}. If R is equal to one of the sets V_{13}, V_{24} or V_{1234} then, since H is clutter, any hyperedge $e \in E$ such that $|d(e)| = 2$,either $e = g$ or $e = h$ where $d(h) = \{x_2, x_4\}$ or $e = f_j, 1 \leq j \leq 4$.

Suppose first $R = \{x\}$ and E contains at most the hyperedge $f = \{x_1, x_2, x_4\}$. Since $\{x\} = V_{1234}$ and $f' = \{x_3, x_2, x_4\} \notin E$ then $(\{x_3, x_2, x_4\}, \{x_3, x\})$ is a 2-coloring of H.

Suppose now $|R| \geq 2$. If $R = V_{13}$ then for any $x \in R$, $C = (\{x_1, x_2, x\}, \{x_3, x_4\} \cup R - \{x\})$ is a 2-coloring of H. If $R = V_{24}$ then for any $x \in R$, $C' = (\{x_1, x_4, x\}, \{x_2, x_3\} \cup R - \{x\})$ is a 2-coloring of H. If $R = V_{1234}$ then C or C' is a 2-coloring of H.

Suppose that $R \neq V_{1234}, R \neq V_{13}$ and $R \neq V_{24}$. We claim that $C = (\{x_1, x_2, x_4\}, \{x_3\} \cup R)$ or $C' = (\{x_3, x_2, x_4\}, \{x_1\} \cup R)$ is a 2-coloring of H.

Note that, since H is clutter, E does not contain a hyperedge $f'_i = \{x_i, x_{i+1}\}, 1 \leq i \leq 4$, distinct of f_i. Since $R \neq V_{1234}$, by Lemma 3, if E contains the hyperedge f with $d(f) = \{x_1, x_2, x_4\}$ or the hyperedge f' with $d(f) = \{x_3, x_2, x_4\}$ then $f \cap R \neq \emptyset$ and $f' \cap R \neq \emptyset$. By supposition, if E contains the hyperedge h with $d(h) = \{x_2, x_4\}$ then $h \cap R \neq \emptyset$.

If for every $1 \leq j \leq 4, f_j \cap R \neq \emptyset$ then C or C' is a 2-coloring of H.

Suppose $f_1 \cap R = \emptyset$ (resp. $f_4 \cap R = \emptyset$). By condition 1, $f_2 \cap R \neq \emptyset$ (resp. by condition 2, $f_3 \cap R \neq \emptyset$), since $R \neq V_{24}$ (resp $R \neq V_{13}$), $f_3 \cap R \neq \emptyset$ (resp. $f_1 \cap R \neq \emptyset$), So C' is a 2-coloring of H. In analogue argument, if $f_2 \cap R = \emptyset$ or $f_3 \cap R = \emptyset$ then C is a 2-coloring of H. □

In analogue way, the following theorem is hold.

Theorem 3 *Suppose* E *contains* $h = \{x_2, x_4\}$ *and does not contain* $g = \{x_1, x_3\}$. H *is 2-colorable if and only if the following conditions hold:*

1) $f_2 \cap R \neq \emptyset$ *or* $f_3 \cap R \neq \emptyset$.
2) $f_1 \cap R \neq \emptyset$ *or* $f_4 \cap R \neq \emptyset$.
3) *If* $R = \{x\}$ *then*
   a) $\{x\} = V_{1234}$.
   b) *there is at least one* $f \notin E$ *such that* $|f| = |d(f)| = 3$.

Theorem 4 *Suppose* E *contains both* $g = \{x_1, x_3\}$ *and* $h = \{x_2, x_4\}$. H *is 2-colorable if and only if one of the following conditions holds:*

1) $|V_{13}| + |V_{1234}| \geq 2$
2) $|V_{24}| + |V_{1234}| \geq 2$

Proof By Corollary 3, $R = V_{13} \cup V_{24} \cup V_{1234}$. Let (A, B) be a 2-coloring of H. Since $g = \{x_1, x_3\}$, $h = \{x_2, x_4\} \in E$, we can suppose without loss of generality that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. If conditions 1 and 2 are not hold then either $|R| = 1$ or $|V_{13}| = |V_{24}| = 1$. In all cases, either $f_1 \cap B = \emptyset$ or $f_3 \cap A = \emptyset$, contradiction.

Suppose condition 1 or 2 is hold. We claim that if $f \in E$ such that $|d(f)| = 2$ then $f \in \{f_1, f_2, f_3, f_4, g, h\}$.

Let $f \in E$ such that $|d(f)| = 2$ and $f \notin \{f_1, f_2, f_3, f_4, g, h\}$. Since H is clutter and $g, h \in E$, $d(f) \neq \{x_1, x_3\}$ and $d(f) \neq \{x_2, x_4\}$. So $d(f) = d(f_j) = \{x_j, x_{j+1}\}, 1 \leq j \leq 4$. Suppose $j = 1$ or 3. Since H is clutter, there are $x \in f - f_j$ and $y \in f_j - f$. As $j = 1$ or 3 and $x \notin f_j$ then $x \in V_{24}$. As $y \in f_j$ and $j = 1$ or 3 then $y \in V_{13} \cup V_{1234}$. Now, $xfx_{j+1}f_jyf_{j+2}x_{j+2}g \cong P_8$, contradiction. Similarly, if $j = 2$ or 4, we could find a P_8.

By this claim, if condition 1 is hold then, for any $x \in V_{13} \cup V_{1234}$ $(\{x_1, x_2, x\}, \{x_3, x_4\} \cup R - \{x\})$ is a 2-coloring of H. If condition 2 is hold then, for any $x \in V_{24} \cup V_{1234}$, $(\{x_1, x_4, x\}, \{x_2, x_3\} \cup R - \{x\})$ is a 2-coloring of H. □

Theorem 5 *Suppose E does not contain $h = \{x_2, x_4\}$ nor $g = \{x_1, x_3\}$. Let $g_1, \dots, g_k, h_1, \dots, h_r \in E$ such that for $1 \leq i \leq k$ and for $1 \leq j \leq r$ $d(g_i) = \{x_1, x_3\}$ and $d(h_j) = \{x_2, x_4\}$. H is 2-colorable if and only if one of the following conditions holds:*

1) $|R| \geq 2$.
2) *If $R = \{x\}$ then*
   a) for some $1 \leq j \leq 4$, $\{x\} = V_{jj+1j+2}$ or $\{x\} = V_{1234}$.
   b) *There is at least one $f \notin E$ such that $|f| = |d(f)| = 3$.*

Proof Suppose H is 2-colorable, let (A, B) be a 2-coloring of H and $R = \{x\}$, then $k = r = 1$. By Corollary 4, either $\{x\} = \vec{V}_{13}$ or $\{x\} = \vec{V}_{24}$ or for some $1 \leq j \leq 4$, $\{x\} = V_{jj+1j+2}$ or $\{x\} = V_{1234}$. If $\{x\} = \vec{V}_{13}$ then $x \notin f_2$ and $x \notin f_4$. So, we can suppose without less of generality that $x_2 \in A, x_3 \in B$ and $x_1 \in A, x_4 \in B$. If $x \in A$ then $f_1 \cap B = \emptyset$, if $x \in B$ then $f_3 \cap A = \emptyset$, contradiction. Similarly, $\{x\} \neq \vec{V}_{24}$. So, either for some $1 \leq j \leq 4$, $\{x\} = V_{jj+1j+2}$ or $\{x\} = V_{1234}$.

Suppose that for some $1 \leq j \leq 4, \{x\} = V_{jj+1j+2}$ and $x \in A$. Since $x \notin f_{j+3}$, we can suppose that $x_{j+3} \in A$ and $x_j \in B$. Let $f \in E$ such that $|f| = |d(f)| = 3$. Since H is clutter and $x \notin f_{j+3}$, $f = \{x_{j+3}, x_{j+1}, x_{j+2}\}$ or $f = \{x_j, x_{j+1}, x_{j+2}\}$. Suppose E contains both $f = \{x_{j+3}, x_{j+1}, x_{j+2}\}$ and $f' = \{x_j, x_{j+1}, x_{j+2}\}$. Since $x \notin f$, $x_{j+1} \in A$ or $x_{j+2} \in B$. If $x_{j+1} \in B$ and $x_{j+2} \in A$ then $f_{j+2} \cap B = \emptyset$, if $x_{j+1} \in A$ and $x_{j+2} \in B$ then $e \cap B = \emptyset$ where $e = g$ or $e = h$, contradiction. So, $x_{j+1}, x_{j+2} \in B$. Now, $f' \cap A = \emptyset$, Contradiction.

Suppose $\{x\} = V_{1234}$ and $x \in A$. Note that, A contains at most one dominated vertex x_j, $1 \leq j \leq 4$, otherwise , A contains a dominated hyperedge f_j, $1 \leq j \leq 4$ or the hyperedge g or h that cannot intersects with B. If for some $1 \leq j \leq 4, x_j \in A$ then $B = \{x_{j+1}, x_{j+2}, x_{j+3}\}$. In this case, $f = \{x_{j+1}, x_{j+2}, x_{j+3}\} \notin E$. If $A = \{x\}$ then $B = \{x_1, x_2, x_3, x_4\}$. In this case E cannot contain any hyperedge f with $|f| = |d(f)| = 3$.

The inverse. If $R = \{x\} = V_{1234}$ then, by condition 3 there is at most one hyperedge $f = \{x_j, x_{j+1}, x_{j+2}\} \notin E$, so, $(\{x_j, x_{j+1}, x_{j+2}\}, \{x_{j+3}, x\})$ is a 2-coloring of H. If $\{x\} = V_{jj+1j+2}$ then $x \notin f_{j+3}$. Since H is clutter and by condition 3, E contains at most one of the two hyperedges $f = \{x_{j+3}, x_{j+1}, x_{j+2}\}$ or $f = \{x_j, x_{j+1}, x_{j+2}\}$. So, $(\{x_j, x_{j+1}, x_{j+2}\}, \{x_{j+3}, x\})$ or $(\{x_{j+3}, x_{j+1}, x_{j+2}\}, \{x_j, x\})$ is a 2-coloring of H.

Suppose $|R| \geq 2$. We will construct a 2-coloring of H. Let $R_1 = \bigcup_{i=1}^{k} g_i - \{x_1, x_3\}$ and $R_2 = \bigcup_{j=1}^{r} h_j - \{x_2, x_4\}$. For our purpose, we distinguish two cases:

Case 1 There is $x \in R_1 \cup R_2$ such that for some $1 \leq i, j \leq 4, , x \in V_{ij}$. By Corollary 4, $x \in \bigcap_{i=1}^{k} g_i - \{x_1, x_3\}$ or $x \in \bigcap_{j=1}^{r} h_j - \{x_2, x_4\}$. Without loss of generality, suppose that $x \in \bigcap_{i=1}^{k} g_i - \{x_1, x_3\}$, then $x \in \vec{V}_{13} \cup \vec{V}_{24} \cup V_{23} \cup V_{14}$. Since $|R| \geq 2$, there is $y \in R$, $y \neq x$. We distinguish two sub-cases:

1.1 There is $y \in R$ such that $y \notin V_{ij}$. By Corollary 4, $y \in V_{st} \cup V_{ll+1l+2} \cup V_{1234}$, for some $1 \leq s, t, l \leq 4$ and $s \neq i$ or $t \neq j$. So, there is at most one dominated hyperedge $f_j, 1 \leq j \leq 4$ with $f_j \cap R = \emptyset$. We claim that $C = (\{x_1, x_2, x_3\}, \{x_4\} \cup R)$ or $C' = (\{x_1, x_3, x_4\}, \{x_2\} \cup R)$ is a 2-coloring of H. By Lemma 4, if $f \in E$ with $d(f) = \{x_1, x_2, x_3\}$ or $d(f) = \{x_1, x_3, x_4\}$ then, $x \in f$, so $f \cap B \neq \emptyset$. Since H is clutter, there no hyperedge $e \in E$ with $e = d(f_j), 1 \leq j \leq 4$. Now, if for every $1 \leq j \leq 4$, $f_j \cap R \neq \emptyset$ then, C or C' is a 2-coloring of H. If $f_1 \cap R = \emptyset$ then $x \in \vec{V}_{24} \cup V_{23}$, so C' is a 2-coloring of H. If $f_2 \cap R = \emptyset$ then, $x \in \vec{V}_{13} \cup V_{14}$, so C' also is a 2-coloring of H. If $f_3 \cap R = \emptyset$ then $x \in \vec{V}_{24} \cup V_{14}$, so C is a 2-coloring of H. If $f_4 \cap R = \emptyset$ then $x \in \vec{V}_{13} \cup V_{23}$, so C also is a 2-coloring of H.

1.2 If for every $y \in R, y \in V_{ij}$, that is $R = V_{ij}$. Since $r, k \geq 1$, then by Corollary 4, $i = 1, j = 3$ or $i = 2, j = 4$ and $R_1 = \bigcap_{i=1}^{k} g_i - \{x_1, x_3\}) = R_2 = (\bigcap_{j=1}^{r} h_j - \{x_2, x_4\})$. Since$|R| \geq 2$, for every $x \in R$, $(\{x_1, x_2, x\}), \{x_3, x_4\} \cup R - \{x\})$ is a 2-coloring of H if $i = 1, j = 3$ and $(\{x_2, x_3, x\}), \{x_1, x_4\} \cup R - \{x\})$ is a 2-coloring of H if $i = 2, j = 4$.

Case 2 For every $x \in R_1 \cup R_2$, $x \notin V_{ij}$ for any $1 \leq i, j \leq 4$. By Corollary 4, $R_1 \cup R_2 \subseteq \bigcup_{j=1}^{4} V_{jj+1j+2} \cup V_{1234}$, and $R = \overleftarrow{V}_{13} \cup \overleftarrow{V}_{24} \cup \bigcup_{j=1}^{4} V_{jj+1j+2} \cup V_{1234}$. We distinguish two sub-cases:

1.1 There is $x \in R_1 \cup R_2$ such that $x \in V_{jj+1j+2}$ for some $1 \leq j \leq 4$. We claim that $V_{jj+1j+2} \subseteq f$

where $d(f) = \{x_j, x_{j+1}, x_{j+3}\}$. Otherwise, if $x \in V_{jj+1j+2}$ and $x \notin f$ then, $f_{j+3}x_{j+3}fx_{j+1}f_jxex_{j+2} \cong P_8$ where $e = g_1$ if $j = 1,3$ or $e = h_1$ if $j = 2,4$, contradiction.

Since $|R| \geq 2$, there is $y \in R, y \neq x$. If there is $y \in R$ and $y \notin V_{jj+1j+2}$ then, $y \in \overleftarrow{V}_{13} \cup \overleftarrow{V}_{24} \cup \bigcup_{j=1}^{4} V_{jj+1j+2} \cup V_{1234}$. If $y \in V_{1234}$ then, for any $1 \leq j \leq 4$, $f_j \cap R \neq \emptyset$. By the above claim and since H is clutter, $(\{x_j, x_{j+1}, x_{j+3}\}, \{x_{j+4}\} \cup R)$ is a 2-coloring of H. If $y \in \overleftarrow{V}_{13} \cup \overleftarrow{V}_{24}$ then there is at most one dominated hyperedge $f_j, 1 \leq j \leq 4$ with $f_j \cap R = \emptyset$. By Lemma 3 and Lemma 4, $V_{13} \cup V_{24} \subseteq f$, where $d(f) = \{x_j, x_{j+2}, x_{j+3}\}$. So, since H is clutter, $(\{x_j, x_{j+2}, x_{j+3}\}, \{x_{j+1}\} \cup R)$ is a 2-coloring of H.

If for every $y \in R, y \in V_{jj+1j+2}$ then, by Corollary 4, $R = R_1 = R_2 = V_{jj+1j+2}$

So, for every $x \in R$, $(\{x_1, x_3, x\}, \{x_2, x_4\} \cup R - \{x\})$ is a 2-coloring of H.

2.2 $V_{jj+1j+2} = \emptyset$ for any $1 \leq j \leq 4$. In this case $R = \overleftarrow{V}_{13} \cup \overleftarrow{V}_{24} \cup V_{1234}$. Since $r, k \geq 1$ then, $V_{1234} \neq \emptyset$. If $\overleftarrow{V}_{13} \cup \overleftarrow{V}_{24} \neq \emptyset$ then by Lemma 3 and Lemma 4, $\overleftarrow{V}_{13} \cup \overleftarrow{V}_{24} \subseteq f$ where $d(f) = \{x_j, x_{j+1}, x_{j+2}\}, 1 \leq j \leq 4$. So, since H is clutter, $(\{x_j, x_{j+2}, x_{j+3}\}, \{x_{j+1}\} \cup R)$ is a 2-coloring of H. If $\overleftarrow{V}_{13} \cup \overleftarrow{V}_{24} = \emptyset$ then $R = V_{1234}$. Since H is clutter, for every $x \in R$, $(\{x_1, x_2, x\}, \{x_3, x_4\} \cup R - \{x\})$ is a 2-coloring of H. □

3. Algorithmic Aspects

The discussion in previous section can be summarized algorithmically as following: Given a hypergraph $H = (V, E)$ whose incidence graph $G = (V \cup E, I)$ is P_8-free. Let $|V| = n$ and $|E| = m$. The following algorithm convert H to a clutter hypergraph, that is, it deletes for every pair $e, f \in E$ with $e \subseteq f$ the hyperedge f from H.

```
Algorithm Convert H to a clutter
for i = 1 to m do
    if e_i ≠ ∅ then
        j = 1
        while j ≤ m do
            if e_j ≠ ∅ and i ≠ j then
                if e_i ⊆ e_j then
                    E = E − {e_j}, e_j = ∅
            j=j+1
```

Obviously, the worst case occurs when H is already clutter and the running time in this case is $O(nm^2)$.

Suppose now H is clutter and its incidence graph G is P_8-free. Moreover, we may assume that H is connected, that is, G is connected, otherwise, we just proceed component-wise. Let D be a dominating set of G such that $G[D] \cong C_8$. Camby and Schaudt in [16] show that the computation of such connected dominating set can be done in time $O(n^5(n+m))$. Let $D = \{x_1, f_1, x_2, f_2, x_3, f_3, x_4, f_4\}$ where $X = \{x_1, x_2, x_3, x_4,\} \subseteq V$, $F = \{f_1, f_2, f_3, f_4\} \subseteq E$ and $G[D] = x_1f_1x_2f_2x_3\,f_3x_4f_4x_1 \cong C_8$.

The following algorithm test weather H is 2-colorable or not.

```
Algorithm 2-colorability
E_13 = ∅, E_24 = ∅, E_3 = ∅, R = V − X
for i = 1 to m do
  if d(e_i) = {x_1, x_3} then E_13 = E_13 ∪ {e_i}
  if d(e_i) = {x_2, x_4} then E_24 = E_24 ∪ {e_i}
  if d(e_i) = e_i and |e_i| = 3 then E_3 = E_3 ∪ {e_i}
if E_13 = {e} = {x_1, x_3} and E_24 = {e} = {x_2, x_4} then
    return 2-colorability type 1
else if E_13 = {e} = {x_1, x_3} then
    return 2-colorability type 2
else if E_24 = {e} = {x_2, x_4} then
    return 2-colorability type 3
else return 2-colorability type 4
```

Remark that, if H is of type 2 or 3 then $|E_3| \leq 2$, and if H is of type 4 then $|E_3| \leq 4$.

```
Procedure 2-colorability type 1
V_13 = f_1 ∩ f_3 ∩ R − (f_2 ∩ f_4)
V_24 = f_2 ∩ f_4 ∩ R − (f_1 ∩ f_3)
V_1234 = f_1 ∩ f_2 ∩ f_3 ∩ f_4 ∩ R
if |V_13| + |V_1234| ≥ 2 or |V_24| + |V_1234| ≥ 2 then
    return H is 2-colorable
else return H is not 2-colorable
```

```
Procedure 2-colorability type 2
If |R| ≥ 2 then
    if (f_1 ∩ R ≠ ∅ or f_2 ∩ R ≠ ∅) and (f_3 ∩ R ≠ ∅ or f_4 ∩
    R ≠ ∅) then
        return H is 2-colorable
  else return H is not 2-colorable
 else let R = {x}
    if x ∈ f_1 ∩ f_2 ∩ f_3 ∩ f_4 and |E_3| ≤ 1 then
      return H is 2-colorable
    else return H is not 2-colorable
```

```
Procedure 2-colorability type 3
If |R| ≥ 2 then
    if (f_1 ∩ R ≠ ∅ or f_4 ∩ R ≠ ∅) and (f_2 ∩ R ≠ ∅
     or f_3 ∩ R ≠ ∅) then
        return H is 2-colorable
    else return H is not 2-colorable
else let R = {x}
    if x ∈ f_1 ∩ f_2 ∩ f_3 ∩ f_4 and |E_3| ≤ 1 then
      return H is 2-colorable
    else return H is not 2-colorable
```

```
Procedure 2-colorability type 4
if R = ∅ then
  if E = F then return H is 2-colorable
  else return H is not 2-colorable
```

else if $E_{13} = \emptyset$ or $E_{24} = \emptyset$ then
    return H is 2-colorable
  else if $|R| \geq 2$ return H is 2-colorable
    else let $R = \{x\}$
      if $x \in f_1 \cap f_2 \cap f_3 \cap f_4$ or $x \in f_j \cap f_{j+1} \cap f_{j+2} - f_{j+3}, 1 \leq j \leq 4$ then
        if $|E_3| \leq 3$ then
          return H is 2-colorable
        else return H is not 2-colorable
      else return H is not 2-colorable

Obviously, Procedure 2-colorability type $i, 1 \leq i \leq 4$, run within $O(n)$ time, and Algorithm 2-colorability run within $O(n+m)$ time. As Algorithm Convert H to a clutter run within $O(nm^2)$ time and $n+m \leq nm^2$ then, the running time of testing weather H is 2-colorable or not is $O(nm^2)$.

4. Conclusions

In this paper we solved hypergraph 2-colorability problem when the incidence graph is P_8-free and having a dominating set isomorphic to C_8. By Theorem 1, such incidence graph may have a dominating set D such that $G[D]$ is P_6-free. So, in order to be this problem solvable completely, one should study this last case. From other part, it seems possible that, with more work, one could push our approach to hypergraphs with P_k-free incidence graphs and a dominated set isomorphic to C_k (k is even). However, more interesting would be to know whether there is any k for which hypergraph 2-colorability for hypergraphs with P_k-free incidence graphs is not solvable in polynomial time.

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References

[1] Alon N. and Bregman Z., "Every 8-Uniform 8-Regular Hypergraph Is 2-Colorable," *Graphs and Combinatorics*, vol. 4, pp. 303-306, 1988.

[2] Beck J., "On 3-Chromatic Hypergraphs," *Discrete Mathematics*, vol. 24, no. 2, pp. 127-137, 1978.

[3] Beeri C., Fagin R., Maier D., and Yannakakis M., "On the Desirability of Acyclic Database Schemes," *Journal of the AssoclaUon for Computing Machinery*, vol. 30, no. 3, pp. 479-513, 1983.

[4] Bernstein F., Zurtheorie der trigonometrischereihen. Leipz. Ber., vol. 60, pp. 325–328, 1908.

[5] Camby E. and Schaudt O., "A New Characterization of Pk-Free Graphs," *Algorithmica*, vol. 75, no. 1, pp. 205-217, 2016.

[6] Erdös P. and Lovász L., "Problems and Results on 3-Chromatic Hypergraphs and Somerelated Questions," *Colloquia Mathematica Societatis Janos Bolyai*, vol. 10, pp. 609-627, 1973.

[7] Erdös P., "On A Combinatorial Problem," *Nordisk Matematisk Tidskrift*, vol. 11, no. 40, pp. 5-10, 1963.

[8] Garey M. and Johnson D., *Computers and Intractability: A Guide to the Theory of NP-completeness*, Freeman and Company, 1979.

[9] Henning M. and Yeo A., "2-Colorings in K-Regular K-Uniform Hypergraphs," *European Journal of Combinatorics*, vol. 34, no. 7, pp. 1192-1202, 2013.

[10] Linial N. and Tarsi M., "Deciding Hypergraph 2-Colorability by H-resolution," *Theoretical Computer Science*, vol. 38, pp. 343-347, 1985.

[11] Lovász L., "Coverings and colorings of hypergraphs," in Proceedings of 4th Southeastern Conf. on Combinatorics, Graph Theory and Computing, Util. Math. Publ. Congr. Numerantium, VIII, pp. 3-12, 1973.

[12] Radhakrishnan J. and Srinivasan A., "Improved bounds and algorithms for hypergraph 2-coloring," *Random Structures and Algorithms*, vol. 16, no. 1, pp. 4-32, 2000.

[13] Szymanska E., "The Complexity of 2-Coloring and Strong Coloring in Uniform Hypergraphs of High Minimum Degree," *Discrete Mathematics and Theoretical Computer Science*, vol. 15, no. 2, pp. 121-138, 2013.

[14] van't Hof P. and Paulusma D., "A New Characterization Of P6-Free Graphs," *Discrete Applied Mathematics*, vol. 158, no. 7, pp. 731-740, 2010.

[15] Vishwanathan S., "Note On 2-Coloring Certain K-Uniform Hypergraphs," *Journal of Combinatorial Theory, Series A*, vol. 101, no. 1, pp. 168-172, 2003.

[16] Zhang H., Song L., and Han Z., "Radio Resource Allocation for Device-to-Device Underlay Communication Using Hypergraph Theory," *IEEE Transactions on Wireless Communications*, vol. 15, no. 7, pp. 4852-4861, 2016.

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