Impact of the Angular Spread and Antenna Spacing on the Capacity of Correlated MIMO Fading Channels

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Abstract: *It has been shown that the capacity of a multiple-input multiple-output system increases linearly with the number of antennas, provided that the environment is rich scattering. However, this increase in the capacity is substantially degraded if the multiple input multiple output channels are correlated. In this paper, the capacity of correlated multiple input multiple output fading channel is investigated for laplacian and uniform angular energy distributions that implicitly represent two different scatterer distributions. The effect of the spatial correlation of a uniform circular antenna array is considered. Optimal power allocation is implemented to maximize the capacity. Through extensive Monte Carlo simulations, the results show that multiple input multiple output channel capacity is a function of the angle spread and antenna spacing. These two parameters play a major role in dictating the spatial correlation which in turn affects on the capacity. Large angle spread leads to lower correlation between the antenna elements and consequently higher capacity. The degradation in the capacity can be reduced by increasing the spacing between the antenna elements.*

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1. Introduction

Multiple Input Multiple Output (MIMO) systems promise a substantial improvement in the wireless system capacity, without requiring more bandwidth. The capacity increases linearly with the number of antennas, provided that the environment has sufficient rich scattering [13, 4, 17, 18]. The substantial increase in MIMO channel capacity arises from the fact that a MIMO channel can be decomposed into many parallel independent sub channels. Assuming independent gaussian data symbols are transmitted simultaneously through these sub channels, a substantial increase in the data rate over that of Single Input Single Output (SISO) system is achieved [13, 4].

In practice, however, the sub channels between the pairs of transmit-receive antennas are not independent, but spatially correlated, and consequently a significant degradation in the promised MIMO channel capacity occurs because of this correlation [5]. Spatial correlation occurs due to poor scattering or limited angular spread and insufficient antenna spacing. A fully correlated MIMO channel only offers one subchannel and thus no advantage is provided by the MIMO system. The capacity in the case of uncorrelated rayleigh fading links between antennas has been extensively investigated [4, 17, 18].

Correlated MIMO channel capacity was analyzed in [12] for Uniform Linear Arrays (ULAs) and uniform Angle Of Arrival (AOA) distribution at the Base Station (BS). In [13], the correlation effect on MIMO channel capacity was investigated based on ray-tracing propagation model. In [9], the capacity of $n \times n$ correlated MIMO channel was investigated using a uniform correlation matrix, which neglects the effect of the antenna spacing on the correlation coefficients.

Most of the published work on MIMO channel capacity assumes a uniform AOA distribution and uniform linear arrays. However, measurements in [9] have shown that the AOA distribution is more likely to resemble laplacian distribution.

In this paper, we investigate the effect of spatial correlation on the capacity of a narrowband MIMO fading channel for laplacian and uniform angular energy distributions. A macrocell environment is considered, where the multipath signals generated by the local rich scattering around a Mobile Station (MS) arrive at an elevated BS antenna within a given angular spread or beamwidth. A circular antenna array at the BS is used. Optimum power allocation is performed to maximize the capacity over the equivalent parallel subchannels.

This paper is organized as follows. Section 2 presents the MIMO system model. The propagation scenario is presented in section 3 the theoretic MIMO channel capacity is briefly described in section 4 the correlation functions and the correlated channel model are described in sections 5 and 6 transmission optimization using water-filling algorithm is described in section 7 numerical results with discussions are presented in section 8. The conclusions are drawn in section 9.

2. MIMO System Model

Consider a narrowband MIMO rayleigh fading channel with M transmit and N receive antennas. The channel can be described by a $N \times M$ channel transfer matrix *H*, whose entries are Independent and Identically Distributed (IID) complex gaussian random variables with zero mean and unit variance. The entry h_{ij} of matrix *H* represents the fading coefficient from the j^{th} transmit antenna element to the i^{th} receive antenna element. The channel input-output relationship can be described in base band form as

$$y = Hx + w \quad (1)$$

where $y = (y_1, y_2, \ldots, y_N)^T$ is the $N \times 1$ received signal vector, $x = (x_1, x_2, \ldots, x_M)^T$ is the $M \times 1$ transmitted signal vector with the power constraint $tr(E(xx^H)) \leq P$, $w = (w_1, w_2, \ldots, w_N)^T$ is the $N \times 1$ zero-mean additive white gaussian noise (AWGN) vector with complex entries and covariance matrix $\sigma_w^2 I_N$, where I_N is an $N \times N$ identity matrix. The $(.)^T$, $(.)^H$, $tr(.)$, $E(.)$ denote transposition, conjugate transpose, trace, and expectation, respectively. The fading matrix *H* can be written in terms of its column vectors as

$$\mathrm{H} = [h_1 \;\; h_2 \;\; \cdots \;\; h_M] \quad (2)$$

where $h_j = (h_{1j}, h_{2j}, \cdots, h_{Nj})^T$ is the $N \times 1$ column vector containing the complex gains from transmit antenna j ($j = 1 \ldots M$) to the receive antenna elements.

3. Propagation Model and Circular Array Response Vector

We consider the uplink propagation in a macrocell mobile environment. In such an environment, the BS antenna is typically located above the local clutter and far away from the scatterers. The MS is surrounded by many scatterers distributed according to a given Probability Density Function (PDF), which determines the angular energy distribution $p_\Theta(\theta)$ of the AOA at the BS. Figure 1 shows the geometry of the propagation model, where Δ is the angle spread with respect to the azimuth central AOA θ_0 (representing the mean AOA). In the next subsection, the angular energy distributions that are assumed in this paper are presented.

3.1. Angular Energy Distributions

Many statistical AOA distributions have been reported in the literature [7, 15, 11]. The most commonly used distributions are the uniform and laplacian distributions. The uniform distribution function is given by [11].

$$p_\theta(\theta) = \begin{cases} 1/2\Delta & \theta_0 - \Delta \leq \theta \leq \theta_0 + \Delta \\ 0 & otherwise \end{cases} \quad (3)$$

where 2Δ is the overall angle spread over which the AOA is changing, θ is the azimuth AOA. The laplacian distribution is given by [16]

$$p_\theta(\theta) = \frac{a}{2(1 - e^{-a\pi})} e^{-a|\theta - \theta_0|},$$
$$-\pi + \theta_0 \leq \theta \leq \pi + \theta_0 \quad (4)$$

where a is a decaying factor which is related to the angle spread Δ, i.e., as a increases, the angle spread decreases.

3.2. Impulse Response Vector of Circular Array

Assume a uniform circular antenna array with N elements that are equally spaced, as shown in Figure 1. For a plane wave arriving with azimuth angle θ, the array response vector can be written as

$$a(\theta) = [a_0, a_1, a_2, \ldots, a_{N-1}]^T \quad (5)$$

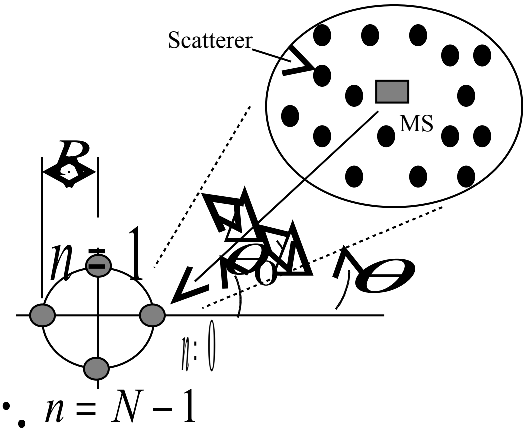

Figure 1. Wireless environment where signals from a mobile arrive at the receiver within $\pm\Delta$ of angle θ_0.

where a_n denotes the channel response at the n[th] element which is positioned at the angle ϕ_n, and is written as

$$a_n = e^{-j\frac{2\pi}{\lambda}R\cos(\theta_l - \phi_n)}, \quad n = 0, 1, \cdots N-1 \quad (6)$$

where R is the radius of the circular array, λ is the wavelength, θ_l is the azimuth AOA of the i^{th} component, and $\phi_n = 2\pi n / N$.

4. Capacity of MIMO Fading Channels

The capacity of MIMO channel is defined as the maximum data rate that can be transmitted over the channel with a probability of error almost close to zero [1]. Assume that the channel matrix *H* is random with entries that are IID, complex gaussian, zero mean, and normalized to have unit variance. The general expression for the ergodic capacity of $N \times M$ MIMO channel is given by [4]

$$C = E_{\mathbf{H}}\left[\log_2 \det(I_N + \frac{\rho}{M} HQH^H)\right] \text{ bits/s/Hz} \quad (7)$$

where $Q = E[\mathrm{xx}^H]$ is the input signal covariance matrix and ρ denotes the Signal-to-Noise Ratio (SNR) at each receive antenna element. The expectation is with respect to the random channel matrix *H,*

When the channel is unknown at the transmitter but perfectly known at the receiver, the optimum choice is to divide the available transmitting power equally between the transmit antenna elements. Assume that the components of the transmitted signal vector *x* are statistically independent (i.e., $Q = I_M$) with Gaussian distribution, then the ergodic capacity in equation 7 reduces to

$$C = \mathrm{E}_{\mathrm{H}}\left[\log_2 \det(I_N + \frac{\rho}{M} HH^H)\right] \text{ bits/s/Hz} \quad (8)$$

In the real world, however, the entries of a MIMO channel matrix are not IID, due to the correlation between the individual subchannels. As a result, the performance of MIMO systems deteriorates and the high spectral efficiency promised by MIMO channel is diminished.

5. Spatial Correlation Functions

The correlation between the elements of the receiver antenna means that the columns of the channel matrix H are independent but the elements of each column are correlated. While, correlation between the elements of the transmitter antenna means that the rows of *H* are independent but the elements of each row are correlated.

Spatial correlation occurs due to insufficient spacing between antenna elements, small angle spread, existence of few dominant scatterers, and the antenna geometry. Assume that the amplitudes of the scattered rays are identical and their phases are completely independent. Further, assume that each scattered ray is reflected only once. Then, the spatial correlation formula between the m^{th} and n^{th} antenna elements is defined as [6].

$$\rho_{mn} = E[a_m(\theta)\, a_n^*(\theta)] = \int_{\theta} a_m(\theta) a_n^*(\theta)\, p_{\theta}(\theta)\, d\theta \quad (9)$$

where $p_{\theta}(\theta)$ is the probability density function of the arriving signal.

For a circular antenna array and when $p_{\theta}(\theta)$ is uniformly distributed as given in equation 3, the real and imaginary parts of the spatial correlation function are expressed as [14]

$$\mathrm{Re}\{\rho_{mn}\} = J_0(Z_c) + 2\sum_{k=1}^{\infty} J_{2k}(Z_c)\cos(2k(\theta_0 + \alpha)) \,.\, \mathrm{sinc}(2k\Delta) \quad (10)$$

$$\mathrm{Im}\{\rho_{mn}\} = 2\sum_{k=0}^{\infty} J_{2k+1}(Z_c)\sin((2k+1)(\theta_0 + \alpha)) \,.\, \mathrm{sinc}((2\mathrm{k}+1)\Delta) \quad (11)$$

where $Z_c = \sqrt{K_1^2 + K_2^2}$, $K_1 = 2\pi \frac{R}{\lambda}[\cos\phi_m - \cos\phi_n]$, $K_2 = 2\pi \frac{R}{\lambda}[\sin\phi_m - \sin\phi_n]$, and α is a relative angle between the m^{th} and n^{th} antenna elements and can be calculated from $\sin\alpha = K_1 / Z_c$ or $\cos\alpha = K_2 / Z_c$. The angle θ_0 is the central AOA and $J_0(z)$ and $J_n(z)$ are the bessel functions of the first kind and 0^{th} and n^{th} order, respectively. When the AOA is distributed according to the laplacian function as given in equation 4, the real and imaginary parts of the spatial correlation between the elements of a circular array are expressed as [8]

$$\mathrm{Re}\{\rho_{mn}\} = J_0(Z_c) + 2\sum_{k=1}^{\infty} J_{2k}(Z_c)\frac{a^2}{a^2 + 4k^2} \,.\, \cos(2k(\theta_0 + \alpha)) \quad (12)$$

$$\mathrm{Im}\{\rho_{mn}\} = -2\sum_{k=0}^{\infty} J_{2k+1}(Z_c)\frac{a^2}{a^2 + (2k+1)^2} \,.\, \frac{1 + \exp(-2\pi)}{1 - \exp(-a\pi)} \sin((2k+1)(\theta_0 + \alpha)) \quad (13)$$

where the variables are defined as before. In the next section, a correlated MIMO channel model is presented.

6. Correlated MIMO Channel Model

Assume that the AOA and Angle Of Departure (AOD) are statistically independent. This allows us to model the spatial correlation at the receiver and transmitter separately [1]. In this case, the correlated channel matrix H_c is modeled as [3]

$$H_C = R_R^{1/2} H_W R_T^{1/2} \tag{14}$$

where H_w is a stochastic $N \times M$ spatially white MIMO channel matrix whose elements are complex Gaussian IID random variables with zero mean and unit variance. The matrices R_R and R_T denote the spatial correlation among the receive and transmit antenna arrays, respectively, and are defined as

$$R_T = E[\mathrm{h}_j h_j^{\mathrm{H}}], \qquad j = 1, \cdots, M \tag{15}$$

$$R_R = E[\mathrm{h}_i h_i^{\mathrm{H}}], \qquad i = 1, \cdots, N \tag{16}$$

where h_J and h_I are the j^{th} row vector and the i^{th} column vector of the uncorrelated channel matrix H, respectively.

Assume rich scattering at the MS, low correlation coefficients between its antennas elements are expected, which can be further reduced by increasing the antenna spacing. This gives us the justification to assume that the spatial correlation at the MS can be neglected. Thus, only the spatial correlation among the entries of the columns of H is considered throughout this paper, i.e., the correlation at the BS. The correlated channel matrix H_c reduces to

$$H_c = R_R^{1/2} H_w \tag{17}$$

The ergodic capacity formula that incorporates the case of correlated receiver elements is given as equation 18, where the variables are defined as before.

$$C = E_H\left[\log_2 \det(I_N + \frac{\rho}{M} R_R^{1/2} H_w H_w^H R_R^{H/2}\right] \text{ bits/s/Hz} \tag{18}$$

7. Optimal Power Allocation

When the channel is known at the transmitter, optimum power allocation can be used to maximize the capacity over Q subject to the constraint $tr(Q) \le P$, where P is the total transmitting power. Optimum method for power allocation in a quasi-static channel is the "water-filling" algorithm [5]. In this method, more power is allocated to the subchannel that has good quality to transmit more information efficiently. To apply this method we need to find the eigenvalues of the matrix HH^H.

From matrix theory, it is well known that any $N \times M$ complex matrix H can be decomposed into a product of unitary matrices and a diagonal matrix by using the Singular Value Decomposition (SVD) as in equation 19, where the unitary matrices U and V are complex $N \times N$ and $M \times M$, respectively.

$$\mathrm{H} = \mathrm{U} \Sigma \mathrm{V}^{\mathrm{H}} \tag{19}$$

$\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_r)$ is an $N \times M$ diagonal matrix with non-negative real elements called the singular values of H and are ordered in a descending manner. Since V is unitary, the matrix $\mathrm{HH^H}$ is written as

$$\mathrm{HH^H} = \mathrm{U} \Sigma \mathrm{V^H} = \mathrm{U} \Sigma \Sigma^{\mathrm{T}} \mathrm{U} \tag{20}$$

let

$$\Lambda = \Sigma\Sigma^T = diag(\sigma_1^2, \sigma_2^2, \cdots, \sigma_r^2) \tag{21}$$

then

$$HH^H U^H = U\Lambda \tag{22}$$

Therefore, the diagonal elements of $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_r)$ are defined as the eigenvalues of HH^H. In order to maximize the capacity C, the optimization problem can be described as maximize C, subject to

$$\sum_{k=1}^{r} p_k = P \tag{23}$$

This problem can be solved by lagrange method, and the solution is

$$p_k = \left(\mu - \frac{\sigma^2}{\lambda_k}\right)^+, \qquad k = 1,2, \cdots, r \tag{24}$$

where $z^+ = max(z, 0)$ and μ is a parameter used for optimizing equation 24, and is determined such that $\sum_{k=1}^{r} p_k = P$ The optimal capacity solution is

$$\mathrm{C} = \sum_{k=1}^{r} \log_2\left(1 + \frac{1}{\sigma^2}(\lambda_k \mu - \sigma^2)^+\right) \text{ bits/s/Hz} \tag{25}$$

8. Numerical Results and Discussions

For simulation purposes, we consider a 4×4 MIMO fading channel. The receive correlation matrix R_R is calculated for the cases of uniform and laplacian energy distributions by using the correlation functions introduced in section 5. The correlated matrix is then generated according to the model in equation 17. Simulations are carried out and each capacity reading at each SNR point is obtained by 10000 random channel realizations. The central AOA θ_0 is assumed to be zero.

The capacity as a function of the uniform angular spread is indicated in Figure 2. It can be seen that with increasing the angle spread, the correlation among the antenna elements decreasing, and the capacity increases. This result is expected, since smaller angular spread leads to higher correlation and consequently lower capacity. The rate of increase in the capacity reduces as the angle spread increases. This effect is due to the decrease in the received SNR as the transmitted energy becomes distributed over a wider angular spread range. At small SNR, the effect of the angular spread on the capacity is not significant.

In Figure 3 the capacity as a function of the laplacian AOA distribution is shown. As the decay factor, a, increases, the correlation between the array elements increases, and the capacity decreases.

Figure 4 shows the ergodic capacity of a correlated 4×4 MIMO fading channel for antenna array radius $R=0.5\lambda$ and when the AOA is uniformly distributed over angular spread range $\Delta=10^{\circ}$. Optimum power allocation is performed based on the water-filling algorithm. The capacity plot when the channel is uncorrelated is included in the Figure for the purposes of comparisons. It can be seen that significant decrease in the capacity of the correlated channel has occurred compared to that of IID rayleigh fading channel. The effect of correlation on the capacity is more significant at higher SNR. This means that as the SNR increases the correlation coefficients between the antenna elements increases and the capacity decrease. For IID channel, the water-filling gain is less significant at higher SNR values, however, slight improvement in the capacity is achieved at lower SNR.

For correlated channel, the capacity gain obtained by optimization is higher than that of IID channel and is constant over the range of SNR values. The impact of laplacian distribution on the capacity is shown in Figure 5 for R = 0.5λ and $a=10$. We note that the rate of the capacity growth versus the SNR reduces faster due to correlation, compared with the case when the AOA follows uniform distribution. The capacity as a function of the antenna spacing for the aplacian distribution is presented in Figure 6. Increasing the antenna radius more than 3λ has no effect on improving the capacity.

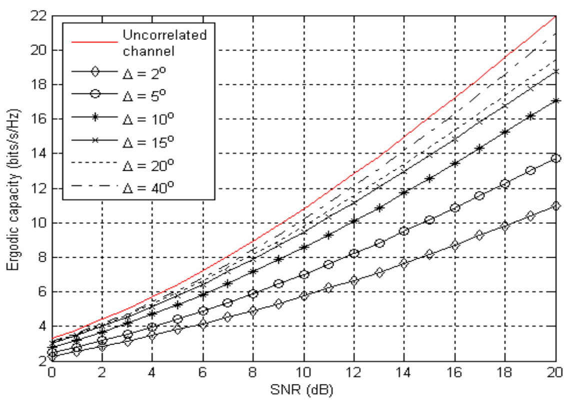

Figure 2. Capacity of a 4x4 correlated MIMO fading channel at different angle spread Δ, uniform distribution, and $R=0.5\lambda$.

9. Conclusions

The capacity of correlated MIMO fading channel has been investigated for uniform and laplacian AOA distributions. Assuming a circular antenna array at the base station, it has been shown that the angular spread and antenna radius have a pronounced effect on the capacity of a correlated MIMO fading channel. Decreasing the angle spread or antenna radius results in increasing the spatial correlation and consequently degradation in the capacity.

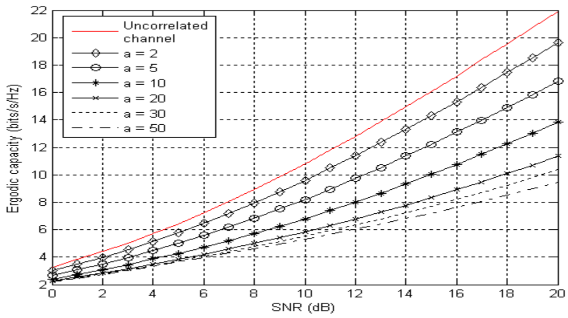

Figure 3. Capacity of a 4x4 correlated MIMO fading channel at different decay values, laplacian distribution and $R=0.5\lambda$.

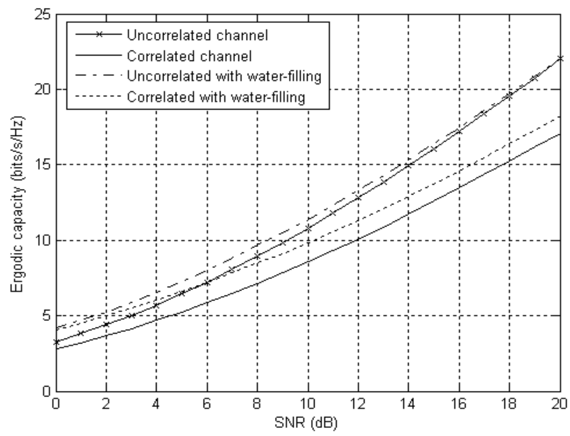

Figure 4. Capacity of a 4x4 correlated MIMO fading channel, with uniform angular distribution, equal and optimized power allocation. $R=0.5\lambda$ and $\Delta=10^{\circ}$.

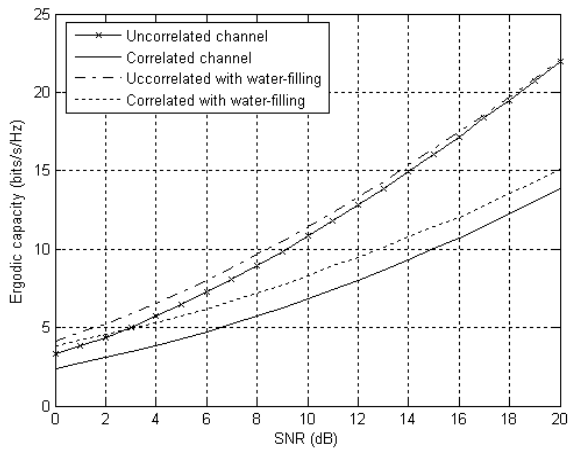

Figure 5. Capacity of a 4x4 correlated MIMO fading channel, with laplacian distribution, equal and optimized power allocation. $R=0.5\lambda$ and α = 10.

Small improvement in the capacity can be achieved by allocating the power optimally to each transmit antenna when the channel parameters are known at the transmitter. Since the angular spread is related to the propagation environment condition, it can not be controlled. Therefore, MIMO channel capacity can be improved by increasing the antenna spacing (radius).

However, increasing the antenna radius more than 3λ, in case of laplacian distribution, has no beneficial effect on the capacity.

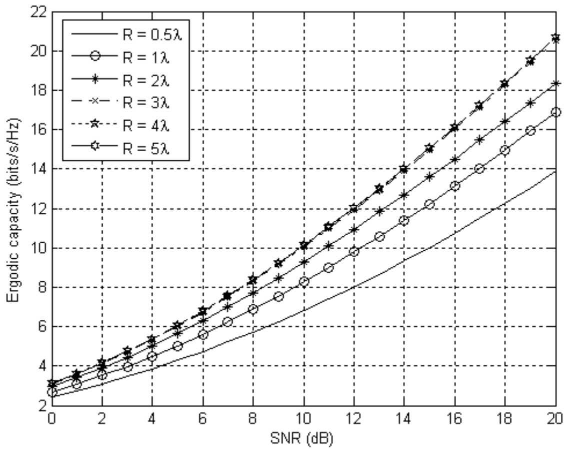

Figure 6. Capacity of a 4x4 correlated MIMO fading channel for different values of R, laplacian distribution, and a=10.

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